Packing Plane Spanning Double Stars into Complete Geometric Graphs

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Abstract

Consider the following problem: Given a complete geometric graph, how many plane spanning trees can be packed into its edge set?

We investigate this question for *plane spanning double stars* instead of general spanning trees. We give a necessary, as well as a sufficient condition for the existence of packings with a given number of plane spanning double stars. We also construct complete geometric graphs with an even number of vertices that cannot be partitioned into plane spanning double stars.

1 Introduction

A *geometric graph* is a drawing of a graph in $\mathbb{R}^2$ where the vertex set is drawn as a point set in general position, that is, no three points lie on a line, and each edge is drawn as a straight-line segment. A geometric graph is called *plane* if no pair of edges crosses. For two vertices v and w in a geometric graph G, we say that *v sees w in G* if the line segment between v and w is not crossed by any edge of G. In this paper we will assume all point sets to be in general position, and for a point set $\mathcal{P}$, we denote by $K(\mathcal{P})$ the complete geometric graph with vertex set $\mathcal{P}$.

It is a long-standing open question whether any complete geometric graph with an even number of vertices has a partition of its edge set into plane spanning trees. If the vertices lie in convex position, the question can be answered in the affirmative, and all possible partitions can be characterized, as was done by Bose et al. [3]. The authors also give a sufficient condition for a complete geometric graph to have a partition of its edge set into plane spanning trees:

Theorem 1 ([3]) *Let $\mathcal{P}$ be a point set with $n = 2m$ points. Suppose that there is a set $\mathcal{L}$ of pairwise non-parallel lines with exactly one point of $\mathcal{P}$ in each open unbounded region formed by $\mathcal{L}$. Then $K(\mathcal{P})$ can be partitioned into m plane spanning trees.*

The trees they construct in this case are double stars: A *double star* is either a single edge or a tree such that the induced subgraph of the vertex set without the leaves is a single edge, called the *spine*.

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More generally, one can ask how many plane spanning trees can be packed into the edge set of a complete geometric graph. Aichholzer et al. [1] show that at least $\lfloor\sqrt{\frac{n}{12}}\rfloor$ plane spanning trees can be packed into any complete geometric graph. This result was very recently improved to $\lceil\frac{n}{3}\rceil$ spanning trees by Garcia [4]. Whereas the trees that Garcia uses for his packing have diameter 4, the trees that Aichholzer et al. construct are again double stars. It seems natural to restrict the open question above to the question whether any complete geometric graph can be partitioned into plane spanning double stars. However, the answer to this question is no. We will show this by proving that for any packing with plane spanning double stars, the spines of the double stars form a matching, which we call the *spine matching*. In the case of a partition, this spine matching is a perfect matching. In Section 3 we then show a necessary condition for a matching to be a spine matching and construct a point set which has no perfect matching that satisfies this necessary condition. In Section 4 we show a sufficient condition for a matching to be a spine matching. Finally, in Section 5 we show that we can decide in polynomial time whether a given matching is a spine matching. Due to space restrictions we cannot give all the proofs. We refer the interested reader to the full version [5].

2 Partitions and Packings

Consider a point set $\mathcal{P}$ of size n and a packing of k plane spanning double stars into $K(\mathcal{P})$. Let M be the set of spines of the double stars.

Lemma 2 *The set of spines M of a packing of k plane spanning double stars into $K(\mathcal{P})$ is a matching.*

As mentioned before, we call this matching the *spine matching*.

Proof. We want to show that no two edges of M are incident. Assume for the sake of contradiction that two edges $e = (p, q)$ and $f = (p, r)$ share an endpoint p. Let E and F be the spanning double stars with spines e and f, respectively. Consider the edge $g = (q, r)$. As all double stars in the partition must be spanning, the point r must be connected to the edge e, which means that $f \in E$ or $g \in E$. As f is already the spine of F, we conclude that $g \in E$. On the other hand q must also be connected to the edge

f and with the same argument we conclude $g \in F$, which is a contradiction. $\square$

Note that for a partition of $K(\mathcal{P})$ into plane spanning double stars, we need $\frac{n}{2}$ double stars, i.e. the spine matching is a perfect matching. We call a perfect matching on a point set $\mathcal{P}$ *expandable* if it is the spine matching of a partition of $K(\mathcal{P})$ into plane spanning double stars.

Corollary 3 *Let $\mathcal{P}$ be a point set that allows a packing of k plane spanning double stars into $K(\mathcal{P})$. Then there is a subset $\mathcal{P}'$ of $\mathcal{P}$ of size $2k$ that allows a partition of $K(\mathcal{P}')$ into plane spanning double stars.*

Proof. Choose $\mathcal{P}'$ as the set of vertices of the spine matching M. $\square$

On the other hand, we can expand a partition on a subset to a packing on the whole point set.

Lemma 4 *Let $\mathcal{P}$ be a point set and let $\mathcal{P}'$ be a subset of $\mathcal{P}$ of size $2k$ that allows a partition of $K(\mathcal{P}')$ into plane spanning double stars. Then $\mathcal{P}$ allows a packing of k plane spanning double stars into $K(\mathcal{P})$.*

For an illustration of the proof see Figure 1

Proof. Consider an edge e in the spine matching M and a point p in $\mathcal{P}\backslash\mathcal{P}'$. Let E be the plane double star with spine $e = (q, r)$ and let $f = (p, q)$ and $g = (p, r)$ be the edges connecting the point p to the spine e. In order to expand E to a plane spanning double star, we have to add either f or g to E without creating a crossing. Assume for the sake of contradiction that both f and g cross an edge of E. Let s and t be the intersections of f and g with E, respectively. Note that the edge of E that crosses f must be incident to r. Similarly, the edge of E that crosses g is incident to q. As q, r, s and t form a convex quadrilateral, we deduce that E is not plane, which is a contradiction. By induction we can therefore expand E to a plane spanning double star. As the spines form a matching we can do this for every double star in the partition of the subset and the claim follows. $\square$

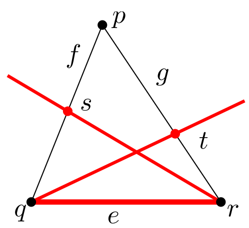

Figure 1: Illustration of the proof of Lemma 4.

Combining Corollary 3 and Lemma 4 we get the following result:

Theorem 5 *Let $\mathcal{P}$ be a point set. Then $K(\mathcal{P})$ allows a packing of k plane spanning double stars if and only if there is a subset $\mathcal{P}'$ of $\mathcal{P}$ of size $2k$ that allows a partition of $K(\mathcal{P}')$ into plane spanning double stars.*

Thus the problem of finding a large packing with plane spanning double stars is equivalent to finding a large subset of the vertex set whose induced graph can be partitioned into plane spanning double stars, i.e. a large expandable matching.

3 A necessary condition

We start by showing that any subset of an expandable matching is again expandable:

Lemma 6 *Let $K(\mathcal{P})$ be partitioned into plane spanning double stars and let $\mathcal{P}'$ be the vertices of any subset of the spine matching M. Then the induced subgraph on $\mathcal{P}'$ inherits a partition into plane spanning double stars.*

Proof. Color each double star in the partition with a different color, including red. Now delete the vertices incident to the red spine and consider the colored subgraph induced by the remaining vertices. This subgraph contains no red edges, as each red edge is incident to the red spine. Also, all deleted edges that are not red cannot be spines. Thus the remaining graph is still partitioned into plane spanning double stars. The result follows by induction. $\square$

Let e be an edge between two points p and q. The *supporting line* ℓ_e of e is the line through p and q.

Let e and f be two edges and let s be the intersection of their supporting lines. If s lies in both e and f, we say that e and f *cross*. If s lies in f but not in e, we say that e *stabs* f and we call the vertex of e that is closer to s the *stabbing vertex* of e. If s lies neither in e nor in f, or even at infinity, we say that e and f are *parallel*. See Figure 2 for an illustration.

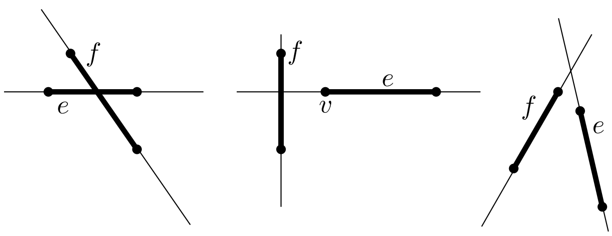

Figure 2: Left: e and f cross; Middle: e stabs f with stabbing vertex v; Right: e and f are parallel.

It can easily be seen that a matching consisting of two parallel edges is not expandable. On the other hand, a matching consisting of two non-parallel edges is expandable, as can be seen in Figure 3.

Lemma 7 *A matching M consisting of two edges e and f is expandable if and only if e and f are not parallel.*

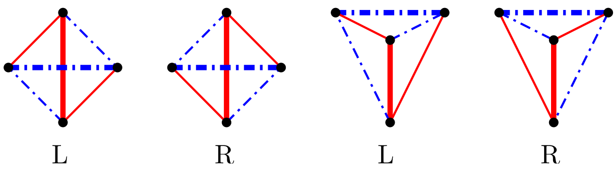

Figure 3: Any pair of crossing or stabbing edges is expandable. The spines are drawn thick.

In Figure 3 we can also see that for two non-parallel edges there are exactly two ways to expand the matching consisting of the two edges into a partition of the induced complete graph. We call them *left-oriented* (L) and *right-oriented* (R).

For larger matchings, the situation is more complicated, but we can still find some configurations that cannot occur in the matching. See Figure 4 for a drawing of these configurations:

A *cross-blocker* is a triple $C = \{e, f, g\}$ of three pairwise non-incident edges such that e and f cross, g stabs both e and f, g does not intersect the convex hull of e and f, and both vertices of g see only one vertex p of e and one vertex q of f in C.

A *stab-blocker* is a triple $S = \{e, f, g\}$ of three pairwise non-incident edges such that f stabs e, g stabs both f and e, g does not intersect the convex hull of e and f, and both vertices of g see only one vertex p of e in S.

Lemma 8 *Let M be a cross-blocker or a stab-blocker. Then M is not expandable.*

For the proof we refer to [5].

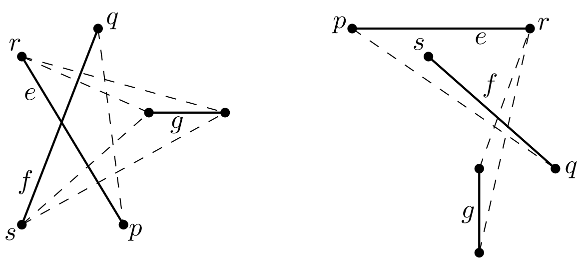

Figure 4: A cross-blocker (left) and a stab-blocker (right).

Combining this with Lemma 7 and Lemma 6, we get a necessary condition for a matching to be expandable.

Theorem 9 *Let $K(\mathcal{P})$ be partitioned into plane spanning double stars. Then the corresponding spine matching M*

- *does not contain two parallel edges,*
- *does not contain a cross-blocker and*
- *does not contain a stab-blocker.*

This allows us to construct a point set whose complete geometric graph cannot be partitioned into plane spanning double stars. For every $k > 0$, we define the *bumpy wheel set* BW_k as follows:

Place $k - 1$ points in convex position and partition them into three sets A_1, A_2, A_3 of consecutive points such that $||A_i| - |A_j|| \leq 1$, $i \neq j$. Let H_i, be the convex hull of $\cup_{j \neq i} A_j$. Place the last point p in the interior such that it lies outside of H_i for all $i \in \{1, 2, 3\}$. See Figure 5 for a depiction of BW_{10}.

It can be shown that every parallel-free perfect matching on BW_k, for $k \geq 10$ even, contains a cross-blocker. Thus no perfect matching on these BW_k is expandable. For k odd there cannot even be a perfect matching.

Theorem 10 *For every $k \geq 9$, the complete geometric graph $K(BW_k)$ cannot be partitioned into plane spanning double stars.*

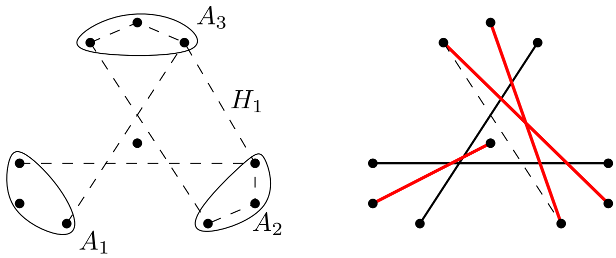

Figure 5: The point set BW_{10} (left) and a cross blocker in a parallel-free matching on this point set (right).

4 A sufficient condition

We will now state a sufficient condition for a matching to be expandable.

A *stabbing chain* are three edges, e, f and g, where e stabs f and f stabs g. We call f the middle edge of the stabbing chain. See Figure 6 for a drawing of some stabbing chains. Note that in the rightmost drawing there are three stabbing chains and each edge is the middle edge in one of the stabbing chains.

Theorem 11 *Let $\mathcal{P}$ be a point set and let M be a perfect matching on $\mathcal{P}$, such that*

(a) *no two edges are parallel,*

(b) *if an edge e stabs two other edges f and g, then the respective stabbing vertices of e lie inside the convex hull of f and g, and*

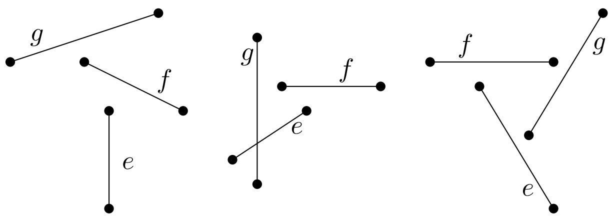

Figure 6: Different stabbing chains.

(c) *if there is a stabbing chain, then the stabbing vertex of the middle edge lies inside the convex hull of the other two edges.*

Then M is expandable.

Note that a stab-blocker is a stabbing chain that satisfies condition (c), but not (b).

We get a partition of $K(\mathcal{P})$ into plane spanning double stars by expanding every pair of edges in M in such a way that the induced K_4 is left-oriented. For a complete proof, we refer to [5]. There it is also shown that the sufficient condition from Bose et al. follows from this result.

Note that this result in particular implies that a set of pairwise crossing edges is expandable. Aronov et al. [2] have shown that every complete geometric graph has a set of at least $\lfloor\sqrt{\frac{n}{12}}\rfloor$ pairwise crossing edges. This proves again the result from Aichholzer et al. [1] that at least $\lfloor\sqrt{\frac{n}{12}}\rfloor$ plane spanning double stars can be packed into any complete geometric graph.

5 Recognizing expandable matchings

In this section we will consider the decision problem where, given a perfect matching on a point set $\mathcal{P}$, we want to decide whether it is expandable. We will show that we can solve this problem in polynomial time.

Recall that there are exactly two ways to expand a pair of non-parallel edges to a partition of their induced K_4 into spanning double stars. We called the two options "left-oriented" and "right-oriented". Expanding a parallel-free perfect matching to a partition into spanning double stars is thus just choosing for each pair of edges in the matching, whether the pair is left-oriented or right-oriented. The given perfect matching is then the spine matching of the partition.

We can check whether a matching of size n is parallel-free by looking at all pairs of edges. As there are $\mathcal{O}(n^2)$ pairs, this can be done in time $\mathcal{O}(n^2)$. If a matching is not parallel-free, it cannot be expandable. So it is enough to only consider parallel-free matchings.

Consider now the partition given by a choice of orientation of each pair of spines in M, where M is parallel-free and has size n, and color each double star with a different color. Assume there is a monochromatic crossing, let us say of color red. Then, as M is parallel-free, the two crossing red edges a and b are incident to exactly three spines: both edges are incident to the red spine e, and each edge is incident to another spine, let us assume that a is incident to the blue spine f, and b is incident to the green spine g. The fact that both a and b are red already determines the orientation of the pairs $\{e, f\}$ and $\{e, g\}$, as a is part of the K_4 induced by e and f and b is part of the K_4 induced by e and g. Also, changing one or both orientations would give a partition where a and b have different colors. Thus each monochromatic crossing can be prevented by forbidding the combination of the orientations of $\{e, f\}$ and $\{e, g\}$ that leads to the crossing being monochromatic. Doing this for every possible monochromatic crossing that could occur in some orientation translates into a 2-CNF with $\mathcal{O}(n^2)$ variables and $\mathcal{O}(n^3)$ clauses, where every variable corresponds to a pair of edges in the matching. For a 2-CNF we can decide whether it is satisfiable in time linear in the number of clauses, so we can decide in time $\mathcal{O}(n^3)$ whether a parallel-free matching is expandable or not. This proves the following theorem:

Theorem 12 *Given a perfect matching M on a point set $\mathcal{P}$ of size n, it is possible to decide in polynomial time whether this perfect matching can be expanded to a partition of $K(\mathcal{P})$ into plane spanning double stars.*

Acknowledgments

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