Reconstructing a Unit-Length Orthogonally Convex Polygon from its Visibility Graph

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Abstract

Reconstructing a polygon from its visibility graph is a fundamental problem, and it is still unknown if simple polygons can be reconstructed in polynomial time. We show that a restricted class of polygons—orthogonally convex polygons with unit-length edges—can be reconstructed from an n-vertex visibility graph in $O(n^4)$ time.

1 Introduction

Recognizing visibility graphs and reconstructing their polygons are fundamental problems that are known to be in PSPACE [3], but it is still unknown if simple polygons can be recognized in polynomial time. Therefore, most research has focused on finding efficient algorithms for restricted classes of polygons, such as spiral [4] and funnel polygons [2]. Surprisingly, very few results exist for even orthogonal polygons: we are only aware of efficient algorithms to recognize convex fans, which consist of a single staircase with an additional vertex [1]. Other algorithms for orthogonal polygons assume extra visibility information is given, such as edge-edge visibility [7, Section 7.3], or "vertical stabs," which capture visibility between vertical edges [6]. See Ghosh's book [5] for a thorough review of results on visibility graphs.

2 Preliminaries

Let P be a polygon on n vertices and edges. We say that two points p and q are *visible* in P if line segment pq is in P. Further, a *visibility graph* $G_P = (V_P, E_P)$ of polygon P has a vertex $v_p \in V_P$ for each vertex p of P, and an edge $(v_p, v_q) \in E_P$ when vertices p and q are visible in P. Edges in G_P that are edges of P are called *boundary edges*. Finally, we note that a maximal clique in G_P corresponds to a maximal convex region whose vertices are a subset of P's vertices.

Unit-Length Orthogonal Polygons. Let P be an orthogonal polygon with unit-length edges, such that no three consecutive vertices on P's boundary are collinear. We call P a *unit-length orthogonal polygon.* We call boundary edges between two convex vertices in a unit-length orthogonal polygon P *tab edges* and their vertices are called *tab vertices.*

We first note that there is a simple algorithm to reconstruct a unit-length orthogonal polygon from its visibility graph if we are given the boundary edges.

Observation 1 *Given a visibility graph $G_P = (V_P, E_P)$ with $n = |V_P|$ vertices and $m = |E_P|$ edges of a unit-length orthogonal polygon P and a Hamiltonian cycle $H = v_0, v_1, \ldots, v_{n-1}$ of the boundary edges of P, we can reconstruct P in $O(n + m)$ time.*

Proof. Omitted. □

In this paper, we consider only unit-length polygons that are *orthogonally convex.* That is, any two points in P are visible via a staircase. We call such polygons unit-length orthogonally convex polygons (UPs). We now focus on finding the boundary edges of a UP, which we can use to reconstruct it by Observation 1.

Properties of UPs. We assume that a UP is axis-aligned, allowing us to use the compass analogy. Note that UPs have four tab edges. We call the tab edge with the largest y-coordinate the north edge, and we similarly name the others the south, east, and west edges. Furthermore, the remaining boundary edges are divided into four staircases, which we refer to as northwest, northeast, southeast, and southwest (i.e., staircases do not contain tab edges).

Note that, for brevity, we only consider polygons with more than 12 vertices. This way, we avoid many special cases in the smaller polygons.

Observation 2 *In a visibility graph of a UP, there is exactly one maximal clique containing all reflex vertices. Moreover, this clique contains no tab vertex.*

Lemma 1 *Every convex vertex u in UP, has a convex neighbor v such that (u, v) is in exactly one maximal clique in the visibility graph of the UP.*

Proof. If u is a tab vertex, then the other tab vertex v is also convex and (u, v) is in exactly one maximal clique. Otherwise, suppose w.l.o.g. that u is on the northwest staircase. Then u has a convex neighbor v on the southeast staircase, and (u, v) is in one maximal clique, which consists of u, v, the reflex vertices within the rectangle R defined by u and v as the opposite corners, and any other corners of R that are convex vertices of the polygon. □

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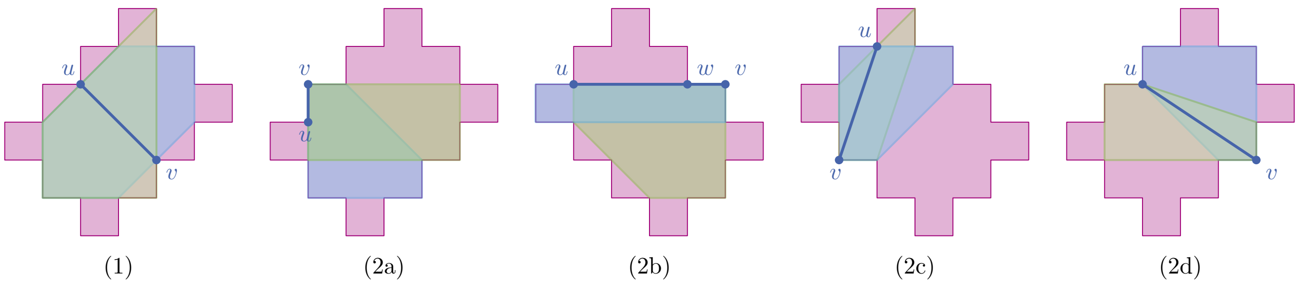

Figure 1: Edges incident to a reflex vertex are contained in at least two maximal cliques.

Lemma 2 *For any edge (u, v) in the visibility graph of a UP, if u or v is reflex, then (u, v) is in at least two maximal cliques. (See Figure 1.)*

Proof. Let $(u, v) \in E_P$ and suppose that at least one of u and v is reflex. Suppose w.l.o.g that u is a reflex vertex on the northwest staircase.

Case 1: Vertex v is reflex. Then u and v are in the maximal clique containing all reflex vertices and no tab vertex, plus u and v see a common tab vertex; therefore (u, v) is in at least two maximal cliques.

For the remaining cases, we assume that v is convex.

Case 2a: Edge (u, v) is a boundary edge. Assume w.l.o.g. that (u, v) is a vertical edge on the northwest staircase and v is above u. If (u, v) is adjacent to the north tab edge, then it is in two maximal cliques: a rectangular clique containing the edge opposite to the tab, and a clique containing an east tab vertex. Suppose (u, v) is not adjacent to the north tab edge. Then u and v see at least 2 convex vertices on the southeast staircase. Thus there are at least two maximal cliques, each containing one of these convex vertices.

We now assume that (u, v) is not a boundary edge.

Case 2b: Edge (u, v) is an axis-aligned visibility edge. We assume w.l.o.g. that (u, v) is a horizontal edge and overlaps boundary edge (v, w). Then v is either on the north- or southeast staircase. If v is on the northeast staircase, u and v are in a maximal clique containing u's boundary neighbor to the west, and one containing u, w, v and all reflex vertex vertices south of w and west of v. If v is on the southeast staircase, then u and v are in a maximal clique containing u's convex boundary neighbor to the north, and another maximal clique containing u, w, v and all reflex vertices north of w and west of v.

Case 2c: Edge (u, v) is a diagonal edge between two adjacent staircases. Then assume w.l.o.g. that v is on the southwest staircase. Then both u and v see a tab vertex t on the north tab. There is a maximal clique containing u, v, t and a maximal clique containing u, v, the reflex vertices north and east of v, and not t.

Case 2d: Edge (u, v) is a diagonal edge between two opposite staircases. Then there is a maximal clique containing u, v, and u's convex boundary neighbor to the west, and another maximal clique containing u, v and u's convex boundary neighbor to the north. ☐

Therefore we can compute all convex vertices, leading to the following lemma.

Lemma 3 *We can identify all convex and reflex vertices in a visibility graph of a UP in $O(n^4)$ time.*

Proof. For each edge, compute if it is in exactly one maximal clique in $O(n^2)$ time. If so, its endvertices are convex. The remaining vertices are reflex. Checking all $O(n^2)$ edges takes $O(n^4)$ time in total. ☐

Definition 1 (regularity) *We call a UP* regular *if each of its staircase boundaries have the same number of vertices. Otherwise, we call it* irregular.

For this extended abstract, we concentrate on irregular unit-length orthogonally convex polygons (IUPs). However, similar methods work for regular polygons.

3 Reconstructing IUPs

From now on, we assume that P is an IUP, and that G_P is its visibility graph. We note that two staircases in an IUP have more vertices than the other two. We call these *long* staircases, and the other ones *short*.

Lemma 4 *An IUP has 4 maximal cliques of size 7 that contain more than 2 convex vertices. Furthermore, each such clique contains exactly one tab edge.*

Proof. Each of the 4 tabs are in exactly one such maximal clique. Other cliques that contain 3 convex vertices have at least 9 vertices, each convex vertex and its two reflex neighbors. ☐

Lemma 5 *In an IUP, we can find a set of at most 8 edges that contains the 4 tab edges in $O(n^4)$ time.*

Proof. Compute the 4 maximal cliques from Lemma 4. These cliques have exactly 3 convex vertices each, and tab edges are incident to two convex vertices, narrowing our choice of tab down to $4 \cdot \binom{3}{2} = 12$ edges. We detect and remove any vertical or horizontal non-boundary edges by checking which edges are in multiple maximal cliques. There are 4 of these; thus, we have 8 edges to consider. ☐

Furthermore these eight edges form 4 disjoint paths on 2 edges, and the middle vertex on each path is a tab vertex. Lastly, we note that these known tab vertices are on the long staircases. We now show how to eliminate the remaining 4 non-tab edges.

Lemma 6 *We can compute the 4 tab edges of an IUP in $O(n^4)$ time.*

Proof. First find the 8 candidates as in Lemma 5.

Recall that we already know one vertex on each tab edge, and that these vertices are on the long staircases. Let one of them be called u. Now it remains the find u's neighbor on its tab edge. Vertex u has two candidate neighbors; let's call them v and w. Just for concreteness, let's say u is on the north edge and is on the northeast staircase.

Suppose w.l.o.g. that v has more reflex neighbors than w, then v is u's neighbor on a tab edge, because it sees reflex vertices on the whole northeast and southeast staircases, while w sees only a subset of those. Otherwise v and w have the same number of reflex neighbors. Then either v or w has more convex neighbors. Suppose w.l.o.g. that v is a tab vertex, then v has fewer convex neighbors than w. To see why, note that since u is on northeast (long) staircase, v is on the northwest (short) staircase. Vertex v has convex neighbors u, w, and every convex vertex on the southeast (short) staircase. Likewise, w has convex neighbors v, u, every convex vertex on the northeast (long) staircase (including u) and one vertex on the southwest (long) staircase.

We can do these checks for all such pairs v and w, giving us all tab edges. □

Now that we have the 4 tab edges, we pick one arbitrarily to be the north edge. We show how to orient the polygon such that the northwest staircase is short and the northeast staircase is long. We do this by identifying the convex vertices on the short staircase by computing *elementary cliques*.

Definition 2 (elementary clique) *An* elementary clique *in an IUP is a maximal clique that contains exactly 3 convex vertices and either contains a tab edge or no tab vertices. (See Figure 2.)*

Lemma 7 *We can identify the elementary cliques with vertices on the northwest staircase in $O(n^4)$ time.*

Proof. First, we compute the $O(n)$ elementary cliques as follows. We compute all axis-aligned, non-boundary visibility edges that have only convex end-vertices. We then compute the 2 maximal cliques containing each such edge, and keep only the cliques with 3 convex vertices.

Let C_0 be the unique maximal (elementary) clique containing the north edge. Then C_0 contains 4 reflex vertices R_0, 3 of which are in exactly one

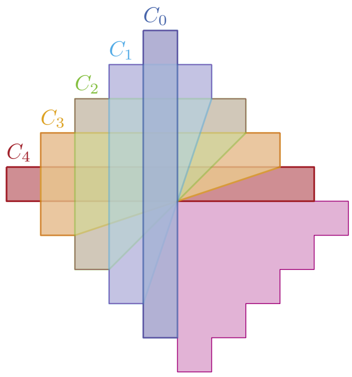

Figure 2: Elementary cliques of an IUP. Note that only half of the elementary cliques are shown.

other elementary clique, which we'll call C_1. The northwest staircase has k_{NW} elementary cliques, $C_0, \ldots, C_{k_{NW}-1}$, where each clique C_i contains reflex vertices R_i. Then $|R_i \cap C_{i+1}| = 3$, and for $j \neq i, i+1$ $R_i \cap C_j = \emptyset$.

Therefore, from elementary clique C_i, we can compute elementary clique C_{i+1} by searching for the only other elementary clique containing reflex vertices R_i. Once we reach an elementary clique containing a tab edge, then we have computed all elementary cliques on the northwest staircase. This tab edge is the west edge and we are finished. □

We now show how to assign the convex vertices from the elementary cliques to each staircase.

Lemma 8 *We can identify the convex vertices on the northwest staircase in $O(n^4)$ time.*

Proof. First we assign all non-tab convex vertices. Let $\mathcal{C}_{\mathrm{NW}}$ be a non-primary elementary clique containing (non-tab) convex vertices v_{NW}, v_{NE} and v_{SW} from the northwest, northeast and southwest staircases respectively. Then v_{NW} sees neither a vertex on the north nor west edge. However, v_{NE} sees a vertex on the west edge, and v_{SW} sees a vertex on the north edge. Therefore, we can compute the non-tab convex vertices on the northwest staircase by checking which vertices have no neighbors on the north or west edges. Now we assign the tab vertices to a staircase. There is exactly one visibility edge connecting a vertex v_N on the north edge to a vertex v_W on the west edge. Then v_W is on the southeast staircase, v_N is on the northeast staircase, and the remaining two tab vertices are on the northwest staircase.

Furthermore, we can assign the remaining convex vertices to the southwest and northeast staircases: Convex vertices on the southwest (northeast) staircase cannot see vertices on the west (north) edge. □

We can repeat the above algorithm to find the convex vertices on the southeast staircase. However, we

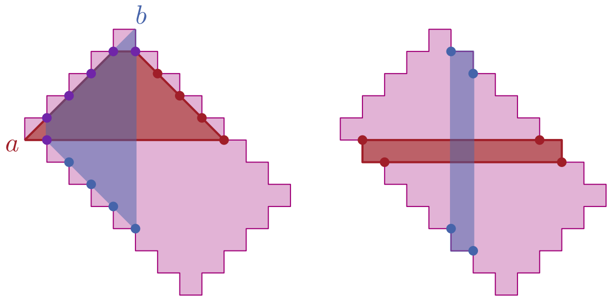

Figure 3: Left: Tab vertices a and b see unique reflex vertices on long staircases. Right: We assign the remaining reflex vertices with rectangles between long staircases.

still need to assign any remaining vertices to the middle of the southwest and northeast (long) staircases. Note that these are the vertices which cannot see the northwest and southeast staircases and, therefore, weren't assigned in the above algorithm.

Lemma 9 *We can assign the convex vertices to their long staircases in $O(n^4)$ time.*

Proof. Let $\mathcal{W}$ and $\mathcal{E}$ be all the convex vertices on the southwest and northeast staircases (which we are computing) and let $\mathcal{W}_0$ and $\mathcal{E}_0$ be the convex vertices on the southwest and northeast staircases that are already known from the elementary cliques from Lemma 8. Let $N_c(v)$ denote the set of convex neighbors on the opposite staircase of some vertex v. Then, for each vertex $w_0 \in \mathcal{W}_0$, $N_c(w_0) \subseteq \mathcal{E}$, i.e., the convex neighbors of the (convex) vertices in $\mathcal{W}_0$ are on the northeast staircase. Similarly, for each vertex $e_0 \in \mathcal{E}_0$, $N_c(e_0) \subseteq \mathcal{W}$. Then we can iteratively define sets $\mathcal{E}_i = \mathcal{E}_{i-1} \setminus \cup_{w \in \mathcal{W}_{i-1}} N_c(w)$ and $\mathcal{W}_i = \mathcal{W}_{i-1} \setminus \cup_{e \in \mathcal{E}_{i-1}} N_c(e)$ and identify all vertices of the southwest and northeast staircases as $\mathcal{W} = \cup_i \mathcal{W}_i$ and $\mathcal{E} = \cup_i \mathcal{E}_i$.

To order the vertices along the southwest staircase, note that the sets $\mathcal{W}_i$ should appear in order of increasing i from top to bottom. Also note that if one were to assign the vertices of W_i the staircase from top to bottom, each vertex w_i in this order would see fewer vertices of $\mathcal{E}_{i-1}$. Thus, we can order the vertices within each $\mathcal{W}_i$. The argument for ordering vertices of $\mathcal{E}_i$ is symmetric. □

We can now choose the south and east edges: a vertex on the east (south) edge can see convex vertices on the southwest (northeast) staircase.

Lemma 10 *We can assign the reflex vertices to each staircase in $O(n^4)$ time.*

Proof. Once the convex vertices are ordered on the staircases, we can compare the reflex vertices that are seen from the tab vertices. Let a and b be vertices on different tab edges, that are visible along a short staircase (see Figure 3, left), and let $\mathcal{R}$ be the set of all reflex vertices of the IUP, and $N(v)$ be a set of all neighbors of vertex v in the visibility graph of IUP. Then $\mathcal{R}_0 = N(a) \cap N(b) \cap \mathcal{R}$ contains all reflex vertices from the short staircase, plus two extra reflex vertices from the neighboring long staircases. The remaining vertices $N(a) \setminus \mathcal{R}_0$ are on one long staircase (and $N(b) \setminus R_0$ are on the other long staircase).

Thus, we can find many reflex vertices on the long staircases, except the end vertices and potentially those in the middle of the staircases. To find the remaining ones, we build rectangles (maximal cliques) consisting of two convex vertices u and v on the opposite staircases and a known reflex vertex w, such that (u, w) forms a boundary edge of the IUP (see Figure 3, right). These rectangles define new reflex vertices on the opposite staircase from w. Thus, we iteratively discover all new reflex vertices. □

Lemma 11 *We can order the reflex vertices on each staircase in $O(n^4)$ time.*

Proof. Let $c_0, \ldots, c_k$ be the convex vertices in order on some staircase S containing reflex vertices R. Then $N(c_i) \cap N(c_{i+1}) \cap R = \{r_i\}$ where r_i is the reflex vertex between c_i and c_{i+1} on staircase S. Thus, we know the order of the reflex vertices along each staircase. □

Therefore, we have ordered the vertices on all staircases, constructing the Hamiltonian cycle of boundary edges in G_P, arriving at the following theorem:

Theorem 12 *In $O(n^4)$ time, we can reconstruct an IUP P from its visibility graph G_P.*

References

[1] J. Abello, Ö. Eğecioğlu, and K. Kumar. Visibility graphs of staircase polygons and the weak Bruhat order, I: From visibility graphs to maximal chains. *Discrete Comput. Geom.*, 14(3):331–358, 1995.

[2] S.-H. Choi, S. Y. Shin, and K.-Y. Chwa. Characterizing and recognizing the visibility graph of a funnel-shaped polygon. *Algorithmica*, 14(1):27–51, 1995.

[3] H. Everett. *Visibility graph recognition.* PhD thesis, University of Toronto, 1990.

[4] H. Everett and D. Corneil. Recognizing visibility graphs of spiral polygons. *J. Algorithms*, 11(1):1–26, 1990.

[5] S. K. Ghosh. *Visibility Algorithms in the Plane.* Cambridge University Press, 2007.

[6] L. Jackson and S. Wismath. Orthogonal polygon reconstruction from stabbing information. *Comp. Geom.-Theor. Appl.*, 23(1):69–83, 2002.

[7] J. O'Rourke. *Art Gallery Theorems and Algorithms.* Oxford University Press, New York, 1987.