One Round Voronoi Game on Grids

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Abstract

Recently there has been a great deal of interest in Voronoi Game: Two players insert a certain number of facilities in a determined number of rounds. The Voronoi Diagram of the inserted facilities is calculated and the winner is settled based on the Voronoi Region occupied by either of the players. A special version of the game in which the players insert their facilities in a single round is called "One Round Voronoi Game". Most of the previous studies in this area are performed in continuous game regions and facilities are considered as single points in the region with no area. In this paper, a new approach to One Round Voronoi Game is presented. Two players insert their facilities on a rectangular grid in one round. The area of the grid is shared between the players based on the *nearest neighbor rule* with *Manhattan* metric. Winning strategies are proposed for the first player in both one and two dimensional grids and the optimality of the strategy is proven in the one-dimensional case. Furthermore, the lower bound of winning margin is presented in both cases.

1 Introduction

Facility location is an optimization problem, concerning with placing a set of facilities which serve a set of customers based on an optimality measure. Adding *competitive market players* to this context and combining it with the arguments of *game theory* leads to the *competitive facility location* problem. This problem has been extensively studied in different fields such as computational geometry, mathematics, industrial engineering and operation research. The *Voronoi game* is a simple geometric model for the competitive facility location problem. From the viewpoint of rounds, there are two types of Voronoi game. In the *one round* version, the first player (White denoted by $\mathscr{W}$) places a set of k facilities in the game region, followed by the second player (Black denoted by $\mathscr{B}$) having the same number of facilities. In the other variation which is called *k-round* game, two players place one facility each alternately for k rounds in the game area.

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Voronoi game has been widely studied in the continuous space domain. One dimensional k-round Voronoi game where the game region is a line segment or a circle was studied by Ahn et al. [1]. The second player ($\mathscr{B}$) always wins the game by a winning margin of arbitrary small $\epsilon > 0$. Their defined k-round game is different from the *one* round game on the continues line segment where $\mathscr{W}$ can achieve a win by placing his facilities at the odd integer points. Also, similar to the k-round case, $\mathscr{W}$ can limit the loss margin as much as he wishes. Fekete and Meijer [2] proposed a model for two dimensional one round game played on a rectangular continuous demand region. They studied the winning conditions in terms of facility count and aspect ratio of the game board. The discrete Voronoi game was introduced by Teramoto et al. [3]. Two players place n facilities each in a graph which contains at least $2n$ nodes. They showed that in a complete $k - ary$ tree which is large enough with respect to n and k, the first player has a winning strategy. The Voronoi game on graphs and particularly on trees were later studied by Kiyomi et al. [4]. They showed that the game played on a path containing n vertexes and continued for $t < \frac{n}{2}$ rounds will end in a tie if either n is even or t is not one. When n is odd and $t = 1$, the first player wins the game. Banik et al. [5] studied another variation of the discrete Voronoi game which is played on a *simple polygon.* They proposed the complexity results when the number of facilities for each player is limited to one. They also studied one round discrete Voronoi game on a line segment [6]. In this problem, the players are competing for owning a set of n users by placing a set of m points each. They proved that if the sorted order of the n points on the line segment is known, the optimal strategy for the second player and first player can be computed in $O(n)$ and $O(n^{m-\lambda_m})$ respectively where $0 < \lambda_m < 1$ is a constant. Gerbner et al. [7] studied t-round voronoi game on graphs. They proved that there are graphs for which the second player gets almost all vertices in the game, but this is not possible for bounded-degree graphs. Further they showed that for trees, the first player can get at least one quarter of the vertices.

In this paper, we study the one round discrete Voronoi game on a grid $G(m, n)$. To achieve a better model, facilities are considered to have area. The problem is studied in one dimensional grid first and a winning strategy that guarantees the winning margin of one is

proposed for $\mathscr{W}$. Further, the optimality of $\mathscr{W}$'s strategy is shown as well. Two dimensional case in which the width of the grid, m, is an odd number is studied as well and condition of $\mathscr{W}$'s win are computed. These computations provide lower bounds in a way that $\mathscr{W}$ wins the game by a margin of m. It is clear that in the grid with even m, the symmetry play by $\mathscr{B}$ ends the game in a tie in most cases. However, proposing a winning strategy for even m seems much harder.
The rest of this paper is organized as follows: In the next section, the game definitions and formulation are presented. In Section 3, Voronoi game on the one dimensional grid is studied. The game in two dimensional grid board is discussed in Section 4. Finally, the last section summarizes some open issues which are introduced by this problem. Please see [8] for the complete proofs.

2 Voronoi Game on Grid

Grid Voronoi Game is denoted by $GVG_r(G,k)$ in which k is the number of facilities for either of the players and r is the number of play rounds. In the rest of this paper $G(m,n)$ is considered as the game play board. G is a rectangular *grid* with the length of n and the width of m and consists of $m \times n$ unit squares called *cell*s. All of the distances are measured using *Manhattan* metric. In the one round game variation ($r = 1$) each of the players (White denoted by $\mathscr{W}$ as the first player and Black denoted by $\mathscr{B}$ as the second player) chooses a set of k facilities disjoint from each other. One or both of the players will own the total area or a part of a cell respectively based on the nearest neighbor rule. Hence, the area of a cell which has the same distance from some cells occupied by $\mathscr{W}$ or $\mathscr{B}$, is shared among them. Furthermore, by placing a facility in a cell, the corresponding player will own all the area of that specific cell. The player owning the largest part of the region is the winner of the game.

3 One Dimensional Grid Voronoi Game

In this section, $G(1,n)$ is considered as a one dimensional grid with the length of n (and the width of $m = 1$). Without the loss of generality, suppose that the orientation of the grid is horizontal as illustrated in Figure 1.

Definition 1 *The distance between two consecutive inserted facilities of $\mathscr{W}$ is called an interval. The horizontal distance between the left side of the game region and the leftmost occupied cell by $\mathscr{W}$ is called left half interval and is denoted by LHI. Likewise, the half interval between the right side of the game region and the rightmost occupied cell by $\mathscr{W}$ is called right half interval and is denoted by RHI. The length of any full/half interval I is denoted by $|I|$.*

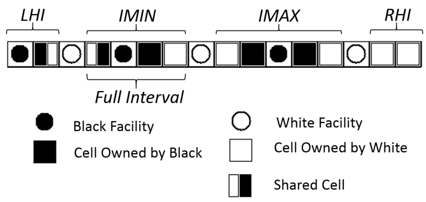


Figure 1: One dimensional grid Voronoi game $GVG_{r=1}(G(m=1,n=16),k=3)$

Considering the definitions we show that selecting the position of facilities according to

$$\left\lfloor \frac{(2i-1)\times n}{2k} \right\rfloor \quad ; i = 1, ..., k \tag{1}$$

in $GVG_1(G(1,n),k)$ is a winning strategy for $\mathscr{W}$. To prove it, the following propositions are required. Note that counting the grid cells is started from zero (see [8] for the extended versions and their proofs).

Proposition 1 *It is obvious that the distance between two optional cells is an integer number. If Eq. (1) is used the maximum length of a full interval in case of existence is $\lfloor \frac{n}{k} \rfloor$. An interval with the maximum length is denoted by IMAX. The minimum length of a full interval in case of existence is $\lfloor \frac{n}{k} \rfloor - 1$. IMIN indicates a full interval with the minimum length. For any n, $|RHI| \leq |LHI|$ holds. As a result $|RHI| + |LHI| \leq \lfloor \frac{n}{k} \rfloor$.*

Proposition 2 *$\mathscr{B}$ will own at least $|LHI|$ of the game region by placing a facility in an IMIN interval. This means that selecting LHI or RHI is dominated by the selection of an empty IMIN interval. Further, Placing two facilities in one IMIN or IMAX interval is not an efficient placing strategy for $\mathscr{B}$.*

Theorem 3 *$\mathscr{W}$ wins $GVG_1(G,k)$ in $G(1,n)$ by selecting the position of his facilities according to Eq. (1) where $2k \nmid n$. The game ends in a tie when $2k \mid n$.*

Proof. Assume that t is the number of IMIN intervals when $\mathscr{W}$ places his facilities according to Eq. (1). The number of IMAX intervals will be $k-1-t$. Considering Propositions 1 and 2, $\mathscr{B}$ is forced to place a facility in each interval and finally places a facility in LHI. Hence, the Voronoi region of $\mathscr{W}$ and $\mathscr{B}$ can be calculated. For the complete proof see [8]. ☐

3.1 Proof of Optimality

In this section, we prove that the placing based on Eq. (1) is an optimal placement strategy for $\mathscr{W}$. It is clear that different arrangements of IMIN and IMAX intervals between LHI and RHI are also optimal placement strategies if placement based on Eq. (1) is optimal. The number of different ways to arrange t objects of one kind (IMAX intervals) and $k-1-t$ objects of another kind (IMIN intervals) in a row (all optimal placement strategies) is $(k-1)!/(t!(k-1-t)!)$. In the following Eq. (1) is used since different arrangements of IMIN and IMAX intervals are equivalent.

Theorem 4 *Placing facilities according to Eq.* (1) *is an optimal placement strategy for* $\mathscr{W}$.

Proof. Suppose that $\mathscr{W}$ uses an arbitrary placement strategy other than Eq. (1) (and its other equivalents). Also, denote the length of created half/full intervals by $L_0, L_1, ..., L_k$ from left to the right side of the grid. It is clear that by inserting a facility in each one of the white intervals, the difference between Voronoi region of $\mathscr{B}$ and $\mathscr{W}$ is

$$\mathrm{MIN}(L_0, L_k) - \mathrm{MAX}(L_0, L_k) + 1. \tag{2}$$

If $\mathrm{MIN}(L_0, L_k) \neq \mathrm{MAX}(L_0, L_k)$ is true, $\mathscr{B}$ will not lose the game (because $\mathrm{MIN}(L_0, L_k) \leq \mathrm{MAX}(L_0, L_k)$). As a result and since $|\mathrm{LHI}| = |\mathrm{RHI}|$ must holds, the loss margin of $\mathscr{B}$ is not more than one if he plays optimally. Also note in previous equations that the length of each interval is at least one. Otherwise, $\mathscr{B}$ always can achieve a tie by following the symmetry play (number of intervals is less than $k+1$). Now, suppose that the length of one of the intervals, I, is bigger than $|\mathrm{IMAX}|$ ($|I| = |\mathrm{IMAX}| + L$). We investigate this problem in two cases :$L \geq 2$ and $L = 1$. First, suppose that $L = 2$. $\mathscr{B}$ gains $\frac{|\mathrm{IMAX}|+3}{2}$ by placing one facility in this interval. Suppose that $L_0 = L_k < \left\lfloor \frac{n}{2k} \right\rfloor$. If $L_0 = L_k > \left\lfloor \frac{n}{2k} \right\rfloor$, placing a facility in an IMIN interval (there exist at least one if $k > 2$) is not efficient, because $|\mathrm{LHI}| = |\mathrm{RHI}| \geq \left\lfloor \frac{n}{2k} \right\rfloor > \frac{1}{2} \left\lfloor \frac{n}{k} \right\rfloor$ and as a result $|\mathrm{LHI}| = |\mathrm{RHI}| = \left\lfloor \frac{n}{2k} \right\rfloor$.
Since $\left\lfloor \frac{n}{2k} \right\rfloor < \frac{|\mathrm{IMAX}|+1}{2}$, placing two facilities in I when $|I| \geq |\mathrm{IMAX}| + 2$ guarantees equality for $\mathscr{B}$. The complete proof can be found in [8]. ☐

4 Two Dimensional Grid Voronoi Game

The game play scenario in two dimensional game is fundamentally different. Both of the players can freely choose the location of their facilities in two directions and as a result the winning strategies will change. Since the facilities in the grid Voronoi game have area, proposing winning strategy is much harder. Furthermore, in the grid Voronoi game more precise winning margin can be calculated and unlike the continuous case, none of the players can limit the loss margin arbitrarily. In the following, the winning condition for $\mathscr{W}$ will be discussed. Note however that, these conditions do not mean that $\mathscr{B}$ wins the game in the rest of cases (unlike the continuous region [2]). It is not difficult to show that $\mathscr{B}$ does not lose the game in the grid with even width (symmetry play in many cases). In this section suppose that $m \geq 3$ is an odd number. We denote the $\left(\frac{m+1}{2}\right)^{th}$ row of the grid by R_{mid} and we call it the *middle row*. Furthermore, similar to the one dimensional case, the horizontal distance between two consecutive facilities of $\mathscr{W}$ (which is a rectangle) is called an interval. In this section, assume that $\mathscr{W}$ will place his facilities according to Eq. (1) horizontally and in R_{mid} vertically. Therefore, the position of every facility of $\mathscr{W}$ is selected based on the following equation:

$$\left(\frac{m+1}{2}, \left\lfloor \frac{(2i-1) \times n}{2k} \right\rfloor\right) \ ; i = 1, ..., k. \tag{3}$$

Lemma 5 *Let* $n_1 = \frac{5}{3}m \times k - \frac{7}{3}k + 1$ *and* $\mathscr{W}$ *places his facilities in* $G(m, n)$ *according to Eq.* (3). *Also, suppose that* $\mathscr{B}$ *has placed a facility in* R_{mid} *in a full interval. For every* $n \geq n_1$, *this position is the most efficient place for the* $\mathscr{B}$*'s facility in that interval.*

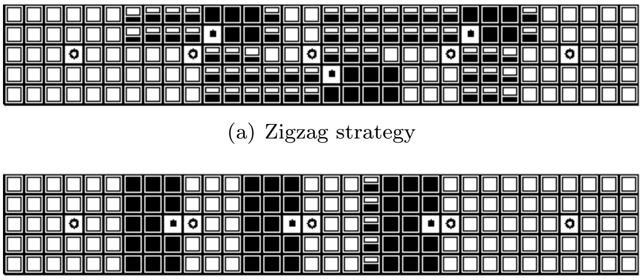
Lemma 6 *Assume that* $\mathscr{B}$ *places a facility in an interval* I *in a way that the total Voronoi region of that facility remains inside the bounds of* I. *Also suppose that the vertical distance from this facility to* R_{mid} *is* $a > 0$. *Transferring this facility vertically to* R_{mid} *will increase the Voronoi region and the amount of increment is* a^2.

Similar calculation for the case when the Voronoi region of a facility is in more than just one interval confirms the result of the previous lemmas. It is obvious now that for any $n \geq n_1$, moving a facility to another cell in the same interval decreases the Voronoi region for the facility (except for R_{mid}). But n_1 is not a tight lower bound (for example $GVG_1(G(7, 29), 3)$). Based on the number of cells which $\frac{1}{3}$ of them are owned by $\mathscr{B}$, it is easy to show that a lower bound for the width of the grid (for win margin of m) can be calculated as follows:

$$n_m = \begin{cases} n_1 & ; \left(\frac{m+1}{2}\right) \bmod 3 = 0 \\ n_1 - (k-2) & ; \left(\frac{m+1}{2}\right) \bmod 3 = 1 \\ n_1 & ; \left(\frac{m+1}{2}\right) \bmod 3 = 2 \end{cases} \tag{4}$$

This equation along with the previous lemmas, decreases the number of possible facility movements to two cases called *valid movements*.

- Transferring a facility from LHI to its neighboring interval (IMIN or IMAX) including the column containing $\mathscr{W}$'s facility.



(a) Zigzag strategy

(b) Simple strategy

Figure 2: Zigzag vs. Simple strategy.

- Transferring a facility from an IMIN interval to a neighboring IMAX one including the column containing $\mathscr{W}$'s facility.

Definition 2 *The intersection of the Voronoi regions of two facilities is called the overlapping of these facilities. Possible cases of overlapping (7 cases) are presented in [8].*

Lemma 7 *Suppose $G(m, n)$ is a grid in which $n \geq n_m$. $\mathscr{W}$ wins $GVG_1(G, 2)$ with the winning margin of m if $2k \nmid n$. The game will end in a tie when $2k \mid n$.*

Lemma 8 *Let $G(m, n)$ be a grid in which $n \geq n_m$. $\mathscr{B}$ loses $GVG_1(G, 3)$ with the minimum loss margin of m if $2k \nmid n$.*

We started to move the facilities by one of the valid movements. Similar calculations indicate that when a movement starts with a valid one it can only continue for at most three facility movements. Theorem 9 covers this problem.

Theorem 9 *For any odd m, any optional k and any $n \geq n_m$, $\mathscr{W}$ wins $GVG_1(G(m, n), k)$ with winning margin of m if $2k \nmid n$.*

Proof. It is clear that if $\mathscr{B}$ plays according to the simple strategy he loses the game by a loss margin of m. We are interested in the possibility of win or a smaller loss margin. To achieve either of these goals consider the first two facilities of $\mathscr{B}$. Assume that the Voronoi region of the first move by $\mathscr{B}$ is P' and the second one is Q'. Also suppose that by placing the same facilities in R_{mid} (according to the simple strategy), $\mathscr{B}$ gains P and Q respectively. It is clear that for a zigzag movement to be efficient, $|P'| + |Q'| > |P| + |Q|$. Considering this, for any k and m in a grid with $n = n_m$ a zigzag movement must start with one of the valid movements and only grows if these conditions hold. One first starts from the leftmost facility of $\mathscr{B}$ and proceeds to the right side of the grid one interval at a time and checks whether one or both of the valid movements are possible. Assume that the first valid movement is possible starting from the left half interval. If $k = 2$ or $k = 3$ by Lemma 7 and Lemma 8 we know that $\mathscr{B}$ loses the game with a loss margin of m. Similar reasoning for $k > 3$ indicates that moving more than three consecutive facilities from R_{mid} starting with a first valid movement and independent of the neighboring intervals type is a non efficient action (Figure 2). Likewise, the second valid movement will become non efficient in at most three moves: the Zigzag movement of just two facilities is not efficient (for $k = 2$ in all cases). Similarly, three movements in all cases are non efficient. $\square$

5 Conclusion and Future Works

An optimal winning strategy for White (the first player) in both one and two dimensional grids is proposed. Like other variations of the Voronoi game problem several questions arise in this context as well. The most interesting problem is probably the case of a grid with even width. Showing that $\mathscr{B}$ does not lose is not difficult. $\mathscr{B}$ can gain at least half of the game region in most cases by *symmetry play* (not possible in all cases). Two dimensional k-round game which is a challenging problem in most contexts is an interesting open problem as well.

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